

CK-12 Probability and Statistics - Advanced (Second Edition)

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Chapter 4

Discrete Probability Distribution

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CHAPTER

4

Discrete Probability Distribution

Chapter Outline

- 4.1 TWO TYPES OF RANDOM VARIABLES
 - 4.2 PROBABILITY DISTRIBUTION FOR A DISCRETE RANDOM VARIABLE
 - 4.3 MEAN AND STANDARD DEVIATION OF DISCRETE RANDOM VARIABLES
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-

4.1 Two Types of Random Variables

Learning Objective

- Learn to distinguish between the two types of random variables: continuous and discrete.

Introduction

The word discrete means countable. For example, the number of students in a class is countable, or discrete. The value could be 2, 24, 34, or 135 students, but it cannot be $\frac{232}{2}$ or 12.23 students. The cost of a loaf of bread is also discrete; it could be \$3.17, for example, where we are counting dollars and cents, but it cannot include fractions of a cent.

On the other hand, if we are measuring the tire pressure in an automobile, we are dealing with a continuous random variable. The air pressure can take values from 0 psi to some large amount that would cause the tire to burst. Another example is the height of your fellow students in your classroom. The values could be anywhere from, say, 4.5 feet to 7.2 feet. In general, quantities such as pressure, height, mass, weight, density, volume, temperature, and distance are examples of continuous random variables. Discrete random variables would usually come from counting, say, the number of chickens in a coop, the number of passing scores on an exam, or the number of voters who showed up to the polls.

Between any two values of a continuous random variable, there are an infinite number of other valid values. This is not the case for discrete random variables, because between any two discrete values, there is an integer number (0, 1, 2, ...) of valid values. Discrete random variables are considered countable values, since you could count a whole number of them. In this chapter, we will only describe and discuss discrete random variables and the aspects that make them important for the study of statistics.

Discrete Random Variables and Continuous Random Variables

In real life, most of our observations are in the form of numerical data that are the observed values of what are called *random variables*. In this chapter, we will study random variables and learn how to find probabilities of specific numerical outcomes.

The number of cars in a parking lot, the average daily rainfall in inches, the number of defective tires in a production line, and the weight in kilograms of an African elephant cub are all examples of *quantitative variables*.

If we let X represent a quantitative variable that can be measured or observed, then we will be interested in finding the numerical value of this quantitative variable. A random variable is a function that maps the elements of the sample space to a set of numbers.

Example: Three voters are asked whether they are in favor of building a charter school in a certain district. Each voter's response is recorded as 'Yes (Y)' or 'No (N)'. What are the random variables that could be of interest in this experiment?

As you may notice, the simple events in this experiment are not numerical in nature, since each outcome is either a 'Yes' or a 'No'. However, one random variable of interest is the number of voters who are in favor of building the

school.

The table below shows all the possible outcomes from a sample of three voters. Notice that we assigned 3 to the first simple event (3 'Yes' votes), 2 to the second (2 'Yes' votes), 1 to the third (1 'Yes' vote), and 0 to the fourth (0 'Yes' votes).

TABLE 4.1:

	Voter #1	Voter #2	Voter #3	Value of Random Variable (number of Yes votes)
1	Y	Y	Y	3
2	Y	Y	N	2
3	Y	N	Y	2
4	N	Y	Y	2
5	Y	N	N	1
6	N	Y	N	1
7	N	N	Y	1
8	N	N	N	0

Figure: Possible outcomes of the random variable in this example from three voters.

In the light of this example, what do we mean by random variable? The adjective 'random' means that the experiment may result in one of several possible values of the variable. For example, if the experiment is to count the number of customers who use the drive-up window in a fast-food restaurant between the hours of 8 AM and 11 AM, the random variable here is the number of customers who drive up within this time interval. This number varies from day to day, depending on random phenomena, such as today's weather, among other things. Thus, we say that the possible values of this random variable range from 0 to the maximum number that the restaurant can handle.

There are two types of random variables—discrete and continuous. Random variables that can assume only a countable number of values are called *discrete*. Random variables that can take on any of the countless number of values in an interval are called *continuous*.

Example: The following are examples of *discrete random variables*:

- The number of cars sold by a car dealer in one month
- The number of students who were protesting the tuition increase last semester
- The number of applicants who have applied for a vacant position at a company
- The number of typographical errors in a rough draft of a book

For each of these, if the variable is X , then $x = 0, 1, 2, 3, \dots$. Note that X can become very large. (In statistics, when we are talking about the random variable itself, we write the variable in uppercase, and when we are talking about the values of the random variable, we write the variable in lowercase.)

Example: The following are examples of *continuous random variables*.

- The length of time it takes a truck driver to go from New York City to Miami
- The depth of drilling to find oil
- The weight of a truck in a truck-weighing station
- The amount of water in a 12-ounce bottle

For each of these, if the variable is X , then $x > 0$ and less than some maximum value possible, but it can take on any value within this range.

Lesson Summary

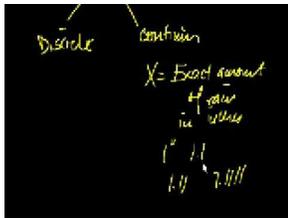
A random variable represents the numerical value of a simple event of an experiment.

Random variables that can assume only a countable number of values are called discrete.

Random variables that can take on any of the countless number of values in an interval are called continuous.

Multimedia Links

For an introduction to random variables and probability distribution functions (3.0), see [khanacademy, Introduction to Random Variables](#) (12:04).



MEDIA

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URL: <http://www.ck12.org/fix/render/embeddedobject/1067>

For examples of discrete and continuous random variables (3.0), see [EducatorVids, Statistics: Random Variables \(Discrete or Continuous\)](#) (1:54).



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4.2 Probability Distribution for a Discrete Random Variable

Learning Objectives

- Know and understand the notion of discrete random variables.
- Learn how to use discrete random variables to solve probabilities of outcomes.

Introduction

In this lesson, you will learn how to construct a probability distribution for a discrete random variable and represent this probability distribution with a graph, a table, or a formula. You will also learn the two conditions that all probability distributions must satisfy.

Probability Distribution for a Discrete Random Variable

The example below illustrates how to specify the possible values that a discrete random variable can assume.

Example: Suppose you simultaneously toss two fair coins. Let X be the number of heads observed. Find the probability associated with each value of the random variable X .

Since there are two coins, and each coin can be either heads or tails, there are four possible outcomes (HH, HT, TH, TT), each with a probability of $\frac{1}{4}$. Since X is the number of heads observed, $x = 0, 1, 2$.

We can identify the probabilities of the simple events associated with each value of X as follows:

$$P(x = 0) = P(TT) = \frac{1}{4}$$

$$P(x = 1) = P(HT) + P(TH) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$P(x = 2) = P(HH) = \frac{1}{4}$$

This is a complete description of all the possible values of the random variable, along with their associated probabilities. We refer to this as a *probability distribution*. This probability distribution can be represented in different ways. Sometimes it is represented in tabular form and sometimes in graphical form. Both forms are shown below.

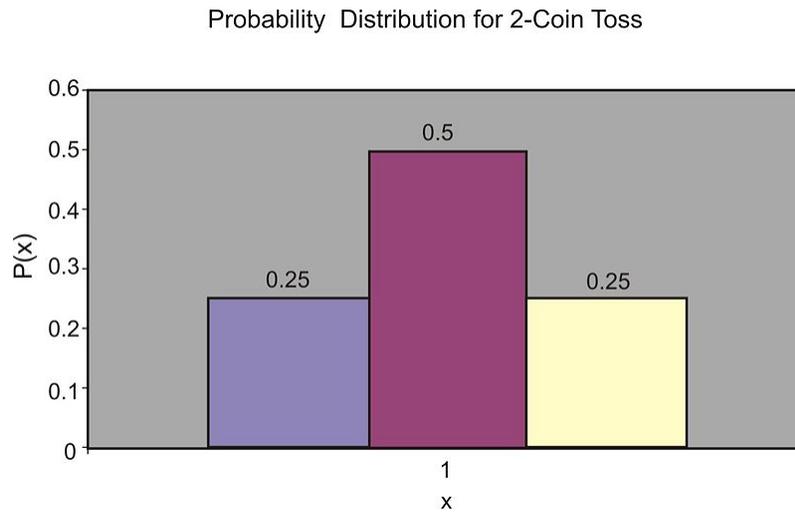
In tabular form:

TABLE 4.2:

x	$P(x)$
0	$\frac{1}{4}$
1	$\frac{1}{2}$
2	$\frac{1}{4}$

Figure: The tabular form of the probability distribution for the random variable in the first example.

As a graph:



A probability distribution of a random variable specifies the values the random variable can assume, along with the probability of it assuming each of these values. All probability distributions must satisfy the following two conditions:

$$P(x) \geq 0, \text{ for all values of } X$$

$$\sum P(x) = 1, \text{ for all values of } X$$

Example: What is the probability distribution for the number of yes votes for three voters? (See the first example in the Chapter Introduction.)

Since each of the 8 outcomes is equally likely, the following table gives the probability of each value of the random variable.

TABLE 4.3:

Value of Random Variable (Number of Yes Votes)	Probability
3	$\frac{1}{8} = 0.125$
2	$\frac{3}{8} = 0.375$
1	$\frac{3}{8} = 0.375$
0	$\frac{1}{8} = 0.125$

Figure: Tabular representation of the probability distribution for the random variable in the first example in the Chapter Introduction.

Lesson Summary

The probability distribution of a discrete random variable is a graph, a table, or a formula that specifies the probability associated with each possible value that the random variable can assume.

All probability distributions must satisfy the following two conditions:

$$P(x \geq 0), \text{ for all values of } X$$

$$\sum P(x) = 1, \text{ for all values of } X$$

Review Questions

1. Consider the following probability distribution:

x	-4	0	1	3
$P(x)$	0.1	0.3	0.4	0.2

- What are all the possible values of X ?
 - What value of X is most likely to happen?
 - What is the probability that $x > 0$?
 - What is the probability that $x = -2$?
2. A fair die is tossed twice, and the up face is recorded each time. Let X be the sum of the up faces.
- Give the probability distribution for X in tabular form.
 - What is $P(x \geq 8)$?
 - What is $P(x < 8)$?
 - What is the probability that x is odd? What is the probability that x is even?
 - What is $P(x = 7)$?
3. If a couple has three children, what is the probability that they have at least one boy?

4.3 Mean and Standard Deviation of Discrete Random Variables

Learning Objectives

- Know the definition of the mean, or expected value, of a discrete random variable.
- Know the definition of the standard deviation of a discrete random variable.
- Know the definition of the variance of a discrete random variable.
- Find the expected value of a variable.

Introduction

In this lesson, you will be presented with the formulas for the mean, variance, and standard deviation of a discrete random variable. You will also be shown many real-world examples of how to use these formulas. In addition, the meaning of expected value will be discussed.

Characteristics of a Probability Distribution

The most important characteristics of any probability distribution are the mean (or average value) and the standard deviation (a measure of how spread out the values are). The example below illustrates how to calculate the mean and the standard deviation of a random variable. A common symbol for the mean is μ (mu), the lowercase m of the Greek alphabet. A common symbol for standard deviation is σ (sigma), the Greek lowercase s .

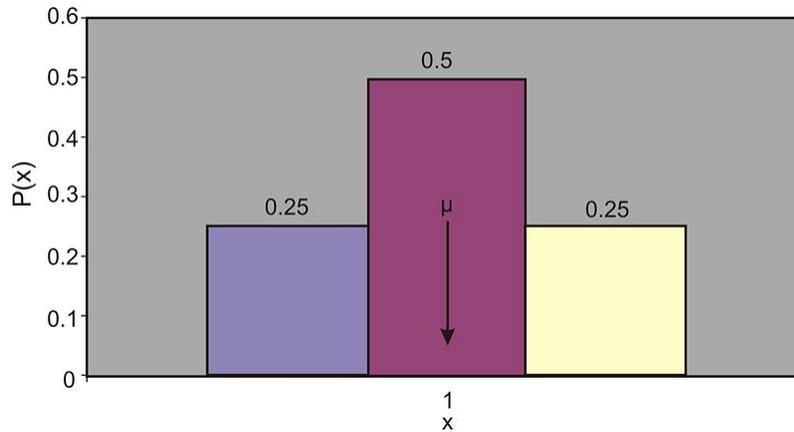
Example: Recall the probability distribution of the 2-coin experiment. Calculate the mean of this distribution.

If we look at the graph of the 2-coin toss experiment (shown below), we can easily reason that the mean value is located right in the middle of the graph, namely, at $x = 1$. This is intuitively true. Here is how we can calculate it:

To calculate the population mean, multiply each possible outcome of the random variable X by its associated probability and then sum over all possible values of X :

$$\mu = (0) \left(\frac{1}{4} \right) + (1) \left(\frac{1}{2} \right) + (2) \left(\frac{1}{4} \right) = 0 + \frac{1}{2} + \frac{1}{2} = 1$$

Probability Distribution for 2-Coin Toss



Mean Value or Expected Value

The mean value, or *expected value*, of a discrete random variable X is given by the following equation:

$$\mu = E(x) = \sum xp(x)$$

This definition is equivalent to the simpler one you have learned before:

$$\mu = \frac{1}{n} \sum_{i=1}^n x_i$$

However, the simpler definition would not be usable for many of the probability distributions in statistics.

Example: An insurance company sells life insurance of \$15,000 for a premium of \$310 per year. Actuarial tables show that the probability of death in the year following the purchase of this policy is 0.1%. What is the expected gain for this type of policy?

There are two simple events here: either the customer will live this year or will die. The probability of death, as given by the problem, is 0.001, and the probability that the customer will live is $1 - 0.001 = 0.999$. The company's expected gain from this policy in the year after the purchase is the random variable, which can have the values shown in the table below.

TABLE 4.4:

Gain, x	Simple Event	Probability
\$310	Live	0.999
-\$14,690	Die	0.001

Figure: Analysis of the possible outcomes of an insurance policy.

Remember, if the customer lives, the company gains \$310 as a profit. If the customer dies, the company "gains" $\$310 - \$15,000 = -\$14,690$, or in other words, it loses \$14,690. Therefore, the expected profit can be calculated as follows:

$$\begin{aligned}\mu &= E(x) = \sum xp(x) \\ \mu &= (310)(99.9\%) + (310 - 15,000)(0.1\%) \\ &= (310)(0.999) + (310 - 15,000)(0.001) \\ &= 309.69 - 14.69 = \$295 \\ \mu &= \$295\end{aligned}$$

This tells us that if the company were to sell a very large number of the 1-year \$15,000 policies to many people, it would make, on average, a profit of \$295 per sale.

Another approach is to calculate the expected payout, not the expected gain:

$$\begin{aligned}\mu &= (0)(99.9\%) + (15,000)(0.1\%) \\ &= 0 + 15 \\ \mu &= \$15\end{aligned}$$

Since the company charges \$310 and expects to pay out \$15, the average profit for the company is \$295 per policy.

Sometimes, we are interested in measuring not just the expected value of a random variable, but also the variability and the central tendency of a probability distribution. To do this, we first need to define population variance, or σ^2 . It is the average of the squared distance of the values of the random variable X from the mean value, μ . The formal definitions of variance and standard deviation are shown below.

The Variance

The variance of a discrete random variable is given by the following formula:

$$\sigma^2 = \sum (x - \mu)^2 P(x)$$

The Standard Deviation

The square root of the variance, or, in other words, the square root of σ^2 , is the standard deviation of a discrete random variable:

$$\sigma = \sqrt{\sigma^2}$$

Example: A university medical research center finds out that treatment of skin cancer by the use of chemotherapy has a success rate of 70%. Suppose five patients are treated with chemotherapy. The probability distribution of x successful cures of the five patients is given in the table below:

x	0	1	2	3	4	5
$p(x)$	0.002	0.029	0.132	0.309	0.360	0.168

Figure: Probability distribution of cancer cures of five patients.

- Find μ .
- Find σ .
- Graph $p(x)$ and explain how μ and σ can be used to describe $p(x)$.
 - To find μ , we use the following formula:

$$\mu = E(x) = \sum xp(x)$$

$$\mu = (0)(0.002) + (1)(0.029) + (2)(0.132) + (3)(0.309) + (4)(0.360) + (5)(0.168)$$

$$\mu = 3.50$$

- To find σ , we first calculate the variance of X :

$$\sigma^2 = \sum (x - \mu)^2 p(x)$$

$$= (0 - 3.5)^2(0.002) + (1 - 3.5)^2(0.029) + (2 - 3.5)^2(0.132)$$

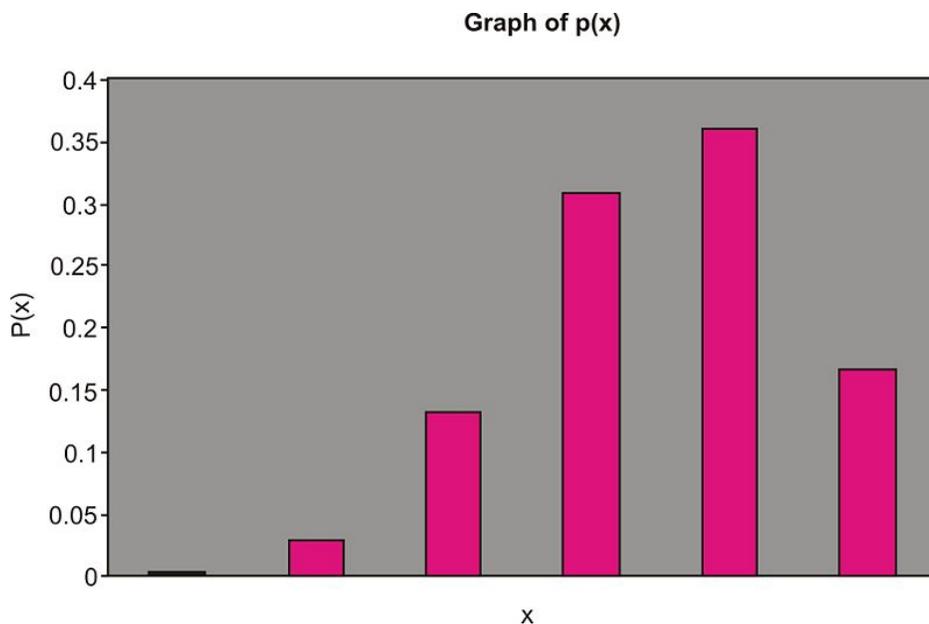
$$+ (3 - 3.5)^2(0.309) + (4 - 3.5)^2(0.36) + (5 - 3.5)^2(0.168)$$

$$\sigma^2 = 1.05$$

Now we calculate the standard deviation:

$$\sigma = \sqrt{\sigma^2} = \sqrt{1.05} = 1.02$$

- The graph of $p(x)$ is shown below:



We can use the mean, or μ , and the standard deviation, or σ , to describe $p(x)$ in the same way we used \bar{x} and s to describe the relative frequency distribution. Notice that $\mu = 3.5$ is the center of the probability distribution. In other

words, if the five cancer patients receive chemotherapy treatment, we expect the number of them who are cured to be near 3.5. The standard deviation, which is $\sigma = 1.02$ in this case, measures the spread of the probability distribution $p(x)$.

Lesson Summary

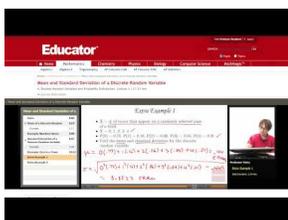
The mean value, or expected value, of the discrete random variable X is given by $\mu = E(x) = \sum xp(x)$.

The variance of the discrete random variable X is given by $\sigma^2 = \sum (x - \mu)^2 p(x)$.

The square root of the variance, or, in other words, the square root of σ^2 , is the standard deviation of a discrete random variable: $\sigma = \sqrt{\sigma^2}$.

Multimedia Links

For an example of finding the mean and standard deviation of discrete random variables **(5.0)(6.0)**, see [Educator Vids, Statistics: Mean and Standard Deviation of a Discrete Random Variable](#) (2:25).

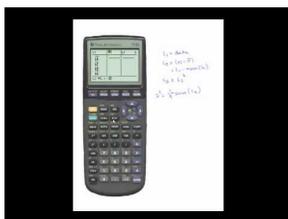


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URL: <http://www.ck12.org/flx/render/embeddedobject/1069>

For a video presentation showing the computation of the variance and standard deviation of a set of data **(11.0)**, see [American Public University, Calculating Variance and Standard Deviation](#) (8:52).

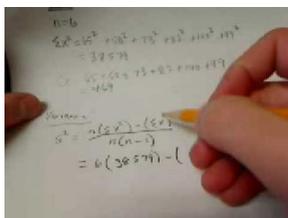


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URL: <http://www.ck12.org/flx/render/embeddedobject/1070>

For an additional video presentation showing the calculation of the variance and standard deviation of a set of data **(11.0)**, see [Calculating Variance and Standard Deviation](#) (4:36).



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Review Questions

1. Consider the following probability distribution:

x	0	1	2	3	4
$p(x)$	0.1	0.4	0.3	0.1	0.1

Figure: The probability distribution for question 1.

- Find the mean of the distribution.
 - Find the variance.
 - Find the standard deviation.
2. An officer at a prison was studying recidivism among the prison inmates. The officer questioned each inmate to find out how many times the inmate had been convicted prior to the inmate's current conviction. The officer came up with the following table that shows the relative frequencies of X , the number of times previously convicted: X , the number of times convicted:

x	0	1	2	3	4
$p(x)$	0.16	0.53	0.20	0.08	0.03

Figure: The probability distribution for question 2. If we regard the relative frequencies as approximate probabilities, what is the expected value of the number of previous convictions of an inmate?

4.4 Sums and Differences of Independent Random Variables

Learning Objectives

- Construct probability distributions of independent random variables.
- Calculate the mean and standard deviation for sums and differences of independent random variables.

Introduction

A probability distribution is the set of values that a random variable can take on. At this time, there are three ways that you can create probability distributions from data. Sometimes previously collected data, relative to the random variable that you are studying, can help to create a probability distribution. In addition to this method, a simulation is also a good way to create an approximate probability distribution. A probability distribution can also be constructed from the basic principles, assumptions, and rules of theoretical probability. The examples in this lesson will lead you to a better understanding of these rules of theoretical probability.

Sums and Differences of Independent Random Variables

Example: Create a table that shows all the possible outcomes when two dice are rolled simultaneously. (Hint: There are 36 possible outcomes.)

TABLE 4.5:

	2^{nd} Die						
	1	2	3	4	5	6	
1	1, 1	1, 2	1, 3	1, 4	1, 5	1, 6	
2	2, 1	2, 2	2, 3	2, 4	2, 5	2, 6	
3	3, 1	3, 2	3, 3	3, 4	3, 5	3, 6	1^{st} Die
4	4, 1	4, 2	4, 3	4, 4	4, 5	4, 6	
5	5, 1	5, 2	5, 3	5, 4	5, 5	5, 6	
6	6, 1	6, 2	6, 3	6, 4	6, 5	6, 6	

This table of possible outcomes when two dice are rolled simultaneously that is shown above can now be used to construct various probability distributions. The first table below displays the probabilities for all the possible sums of the two dice, and the second table shows the probabilities for each of the possible results for the larger of the two numbers produced by the dice.

TABLE 4.6:

Sum of Two Dice, x	Probability, $p(x)$
2	$\frac{1}{36}$
3	$\frac{2}{36}$
4	$\frac{3}{36}$

TABLE 4.6: (continued)

Sum of Two Dice, x	Probability, $p(x)$
5	$\frac{4}{36}$
6	$\frac{5}{36}$
7	$\frac{6}{36}$
8	$\frac{5}{36}$
9	$\frac{4}{36}$
10	$\frac{3}{36}$
11	$\frac{2}{36}$
12	$\frac{1}{36}$
Total	1

TABLE 4.7:

Larger Number, x	Probability, $p(x)$
1	$\frac{1}{36}$
2	$\frac{3}{36}$
3	$\frac{5}{36}$
4	$\frac{7}{36}$
5	$\frac{9}{36}$
6	$\frac{11}{36}$
Total	1

When you roll the two dice, what is the probability that the sum is 4? By looking at the first table above, you can see that the probability is $\frac{3}{36}$.

What is the probability that the larger number is 4? By looking at the second table above, you can see that the probability is $\frac{7}{36}$.

Example: The Regional Hospital has recently opened a new pulmonary unit and has released the following data on the proportion of silicosis cases caused by working in the coal mines. Suppose two silicosis patients are randomly selected from a large population with the disease.

TABLE 4.8:

Silicosis Cases	Proportion
Worked in the mine	0.80
Did not work in the mine	0.20

There are four possible outcomes for the two patients. With 'yes' representing "worked in the mines" and 'no' representing "did not work in the mines", the possibilities are as follows:

TABLE 4.9:

	First Patient	Second Patient
1	No	No
2	Yes	No
3	No	Yes
4	Yes	Yes

As stated previously, the patients for this survey have been randomly selected from a large population, and therefore,

the outcomes are independent. The probability for each outcome can be calculated by multiplying the appropriate proportions as shown:

$$\begin{aligned} P(\text{no for 1}^{\text{st}}) \bullet P(\text{no for 2}^{\text{nd}}) &= (0.2)(0.2) = 0.04 \\ P(\text{yes for 1}^{\text{st}}) \bullet P(\text{no for 2}^{\text{nd}}) &= (0.8)(0.2) = 0.16 \\ P(\text{no for 1}^{\text{st}}) \bullet P(\text{yes for 2}^{\text{nd}}) &= (0.2)(0.8) = 0.16 \\ P(\text{yes for 1}^{\text{st}}) \bullet P(\text{yes for 2}^{\text{nd}}) &= (0.8)(0.8) = 0.64 \end{aligned}$$

If X represents the number silicosis patients who worked in the mines in this random sample, then the first of these outcomes results in $x = 0$, the second and third each result in $x = 1$, and the fourth results in $x = 2$. Because the second and third outcomes are disjoint, their probabilities can be added. The probability distribution for X is given in the table below:

TABLE 4.10:

x	Probability, $p(x)$
0	0.04
1	$0.16 + 0.16 = 0.32$
2	0.64

Expected Values and Standard Deviation

Example: Suppose an individual plays a gambling game where it is possible to lose \$2.00, break even, win \$6.00, or win \$20.00 each time he plays. The probability distribution for each outcome is provided by the following table:

TABLE 4.11:

Winnings, x	Probability, $p(x)$
-\$2	0.30
\$0	0.40
\$6	0.20
\$20	0.10

The table can be used to calculate the expected value and the variance of this distribution:

$$\begin{aligned} \mu &= \sum xp(x) \\ \mu &= (-2 \cdot 0.30) + (0 \cdot 0.40) + (6 \cdot 0.20) + (20 \cdot 0.10) \\ \mu &= 2.6 \end{aligned}$$

Thus, the player can expect to win \$2.60 playing this game.

The variance of this distribution can be calculated as shown:

$$\begin{aligned} \sigma^2 &= \sum (x - \mu)^2 p(x) \\ \sigma^2 &= (-2 - 2.6)^2(0.30) + (0 - 2.6)^2(0.40) + (6 - 2.6)^2(0.20) + (20 - 2.6)^2(0.10) \\ \sigma^2 &\approx 41.64 \\ \sigma &\approx \sqrt{41.64} \approx \$6.45 \end{aligned}$$

Example: The following probability distribution was constructed from the results of a survey at the local university. The random variable is the number of fast food meals purchased by a student during the preceding year (12 months). For this distribution, calculate the expected value and the standard deviation.

TABLE 4.12:

Number of Meals Purchased Within 12 Months, x	Probability, $p(x)$
0	0.04
[1 – 6)	0.30
[6 – 11)	0.29
[11 – 21)	0.17
[21 – 51)	0.15
[51 – 60)	0.05
Total	1.00

You must begin by estimating a mean for each interval, and this can be done by finding the center of each interval. For the first interval of [1 – 6), 6 is not included in the interval, so a value of 3 would be the center. This same procedure can be used to estimate the mean of all the intervals. Therefore, the expected value can be calculated as follows:

$$\begin{aligned}\mu &= \sum xp(x) \\ \mu &= (0)(0.04) + (3)(0.30) + (8)(0.29) + (15.5)(0.17) + (35.5)(0.15) + (55)(0.05) \\ \mu &= 13.93\end{aligned}$$

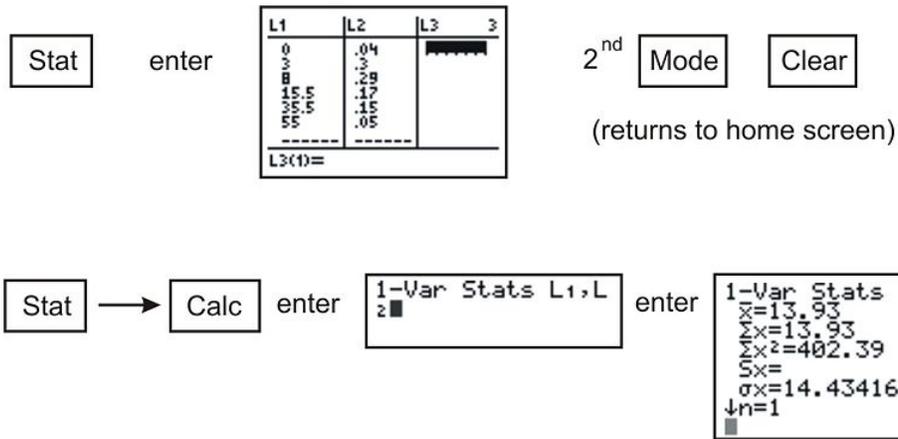
Likewise, the standard deviation can be calculated:

$$\begin{aligned}\sigma^2 &= \sum (x - \mu)^2 p(x) \\ &= (0 - 13.93)^2(0.04) + (3 - 13.93)^2(0.30) \\ &\quad + (8 - 13.93)^2(0.29) + (15.5 - 13.93)^2(0.17) \\ &\quad + (35.5 - 13.93)^2(0.15) + (55 - 13.93)^2(0.05) \\ &\approx 208.3451\end{aligned}$$

$$\sigma \approx 14.43$$

Thus, the expected number of fast food meals purchased by a student at the local university is 13.93, and the standard deviation is 14.43. Note that the mean should not be rounded, since it does not have to be one of the values in the distribution. You should also notice that the standard deviation is very close to the expected value. This means that the distribution will be skewed to the right and have a long tail toward the larger numbers.

Technology Note: *Calculating mean and variance for probability distribution on TI-83/84 Calculator*



Notice that the mean, which is denoted by \bar{x} in this case, is 13.93, and the standard deviation, which is denoted by σ_x , is approximately 14.43.

Linear Transformations of

If you add the same value to all the numbers of a data set, the shape and standard deviation of the data set remain the same, but the value is added to the mean. This is referred to as *re-centering* the data set. Likewise, if you *rescale* the data, or multiply all the data values by the same nonzero number, the basic shape will not change, but the mean and the standard deviation will each be a multiple of this number. (Note that the standard deviation must actually be multiplied by the absolute value of the number.) If you multiply the numbers of a data set by a constant d and then add a constant c , the mean and the standard deviation of the transformed values are expressed as follows:

$$\mu_{c+dX} = c + d\mu_X$$

$$\sigma_{c+dX} = |d|\sigma_X$$

These are called *linear transformations*, and the implications of this can be better understood if you return to the casino example.

Example: The casino has decided to triple the prizes for the game being played. What are the expected winnings for a person who plays one game? What is the standard deviation? Recall that the expected value was \$2.60, and the standard deviation was \$6.45.

Solution:

The simplest way to calculate the expected value of the tripled prize is $(3)(\$2.60)$, or \$7.80, with a standard deviation of $(3)(\$6.45)$, or \$19.35. Here, $c = 0$ and $d = 3$. Another method of calculating the expected value and standard deviation would be to create a new table for the tripled prize:

TABLE 4.13:

Winnings, x	Probability, p
-\$6	0.30
\$0	0.40
\$18	0.20
\$60	0.10

```

1-Var Stats
x̄=7.8
Σx=7.8
Σx²=435.6
Sx=
σx=19.35871897
↓n=1

```

The calculations can be done using the formulas or by using a graphing calculator. Notice that the results are the same either way.

This same problem can be changed again in order to introduce the *Addition Rule* and the *Subtraction Rule* for random variables. Suppose the casino wants to encourage customers to play more, so it begins demanding that customers play the game in sets of three. What are the expected value (total winnings) and standard deviation now?

Let X, Y and Z represent the total winnings on each game played. If this is the case, then μ_{X+Y+Z} is the expected value of the total winnings when three games are played. The expected value of the total winnings for playing one game was \$2.60, so for three games the expected value is:

$$\begin{aligned}\mu_{X+Y+Z} &= \mu_X + \mu_Y + \mu_Z \\ \mu_{X+Y+Z} &= \$2.60 + \$2.60 + \$2.60 \\ \mu_{X+Y+Z} &= \$7.80\end{aligned}$$

Thus, the expected value is the same as that for the tripled prize.

Since the winnings on the three games played are independent, the standard deviation of X, Y and Z can be calculated as shown below:

$$\begin{aligned}\sigma^2_{X+Y+Z} &= \sigma^2_X + \sigma^2_Y + \sigma^2_Z \\ \sigma^2_{X+Y+Z} &= 6.45^2 + 6.45^2 + 6.45^2 \\ \sigma^2_{X+Y+Z} &\approx 124.8075 \\ \sigma_{X+Y+Z} &\approx \sqrt{124.8075} \\ \sigma_{X+Y+Z} &\approx 11.17\end{aligned}$$

This means that the person playing the three games can expect to win \$7.80 with a standard deviation of \$11.17. Note that when the prize was tripled, there was a greater standard deviation (\$19.36) than when the person played three games (\$11.17).

The Addition and Subtraction Rules for random variables are as follows:

If X and Y are random variables, then:

$$\begin{aligned}\mu_{X+Y} &= \mu_X + \mu_Y \\ \mu_{X-Y} &= \mu_X - \mu_Y\end{aligned}$$

If X and Y are independent, then:

$$\sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y$$

$$\sigma^2_{X-Y} = \sigma^2_X + \sigma^2_Y$$

Variations are added for both the sum and difference of two independent random variables, because the variation in each variable contributes to the overall variation in both cases. (Subtracting is the same as adding the opposite.) Suppose you have two dice, one die, X , with the usual positive numbers 1 through 6, and another, Y , with the negative numbers -1 through -6 . Next, suppose you perform two experiments. In the first, you roll the first die, X , and then the second die, Y , and you compute the difference of the two rolls. In the second experiment, you roll the first die and then the second die, and you calculate the sum of the two rolls.

Difference (X-Y)		Sum (X+Y)	
Difference	Probability	Sum	Probability
12	1/36	5	1/36
11	2/36	4	2/36
10	3/36	3	3/36
9	4/36	2	4/36
8	5/36	1	5/36
7	6/36	0	6/36
6	5/36	-1	5/36
5	4/36	-2	4/36
4	3/36	-3	3/36
3	2/36	-4	2/36
2	1/36	-5	1/36

$$\mu_X = \sum xp(x)$$

$$\mu_X = 3.5$$

$$\sigma^2_X \approx \sum (x - \mu_X)^2 p(x)$$

$$\sigma^2_X \approx 2.917$$

$$\mu_{X-Y} = \mu_X - \mu_Y$$

$$\mu_{X-Y} = 3.5 - (-3.5) = 7$$

$$\sigma^2_{X-Y} = \sigma^2_X + \sigma^2_Y$$

$$\sigma^2_{X-Y} \approx 2.917 + 2.917 = 5.834$$

$$\mu_Y = \sum yp(y)$$

$$\mu_Y = -3.5$$

$$\sigma^2_Y \approx \sum (y - \mu_Y)^2 p(y)$$

$$\sigma^2_Y \approx 2.917$$

$$\mu_{X+Y} = \mu_X + \mu_Y$$

$$\mu_{X+Y} = 3.5 + (-3.5) = 0$$

$$\sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y$$

$$\sigma^2_{X+Y} \approx 2.917 + 2.917 = 5.834$$

Notice how the expected values and the variances for the two dice combine in these two experiments.

Example: Beth earns \$25.00 an hour for tutoring but spends \$20.00 an hour for piano lessons. She saves the difference between her earnings for tutoring and the cost of the piano lessons. The numbers of hours she spends on each activity in one week vary independently according to the probability distributions shown below. Determine her expected weekly savings and the standard deviation of these savings.

TABLE 4.14:

Hours of Piano Lessons, x	Probability, $p(x)$
0	0.3
1	0.3
2	0.4

TABLE 4.15:

Hours of Tutoring, y	Probability, $p(y)$
1	0.2
2	0.3
3	0.2
4	0.3

X represents the number of hours per week taking piano lessons, and Y represents the number of hours tutoring per week. The mean and standard deviation for each can be calculated as follows:

$$\begin{aligned}
 E(x) = \mu_X &= \sum xp(x) & \sigma^2_X &= \sum (x - \mu_X)^2 p(x) \\
 \mu_X &= (0)(0.3) + (1)(0.3) + (2)(0.4) & \sigma^2_X &= (0 - 1.1)^2(0.3) + (1 - 1.1)^2(0.3) + (2 - 1.1)^2(0.4) \\
 \mu_X &= 1.1 & \sigma^2_X &= 0.69 \\
 & & \sigma_X &= 0.831
 \end{aligned}$$

$$\begin{aligned}
 E(y) = \mu_Y &= \sum yp(y) & \sigma^2_Y &= \sum (y - \mu_Y)^2 p(y) \\
 \mu_Y &= (1)(0.2) + (2)(0.3) + (3)(0.2) + (4)(0.3) & \sigma^2_Y &= (1 - 2.6)^2(0.2) + (2 - 2.6)^2(0.3) + (3 - 2.6)^2(0.2) \\
 & & &+ (4 - 2.6)^2(0.3) \\
 \mu_Y &= 2.6 & \sigma^2_Y &= 1.24 \\
 & & \sigma_Y &= 1.11
 \end{aligned}$$

The expected number of hours Beth spends on piano lessons is 1.1 with a standard deviation of 0.831 hours. Likewise, the expected number of hours Beth spends tutoring is 2.6 with a standard deviation of 1.11 hours.

Beth spends \$20 for each hour of piano lessons, so her mean weekly cost for piano lessons can be calculated with the *Linear Transformation Rule* as shown:

$$\mu_{20X} = (20)(\mu_X) = (20)(1.1) = \$22 \text{ by the Linear Transformation Rule.}$$

Beth earns \$25 for each hour of tutoring, so her mean weekly earnings from tutoring are as follows:

$$\mu_{25Y} = (25)(\mu_Y) = (25)(2.6) = \$65 \text{ by the Linear Transformation Rule.}$$

Thus, Beth's expected weekly savings are:

$$\mu_{25Y} - \mu_{20X} = \$65 - \$22 = \$43 \text{ by the Subtraction Rule.}$$

The standard deviation of the cost of her piano lessons is:

$$\sigma_{20X} = (20)(0.831) = \$16.62 \text{ by the Linear Transformation Rule.}$$

The standard deviation of her earnings from tutoring is:

$$\sigma_{25Y} = (25)(1.11) = \$27.75 \text{ by the Linear Transformation Rule.}$$

Finally, the variance and standard deviation of her weekly savings is:

$$\begin{aligned}
 \sigma^2_{25Y-20X} &= \sigma^2_{25Y} + \sigma^2_{20X} = (27.75)^2 + (16.62)^2 = 1046.2896 \\
 \sigma_{25Y-20X} &\approx \$32.35
 \end{aligned}$$

Lesson Summary

A chance process can be displayed as a probability distribution that describes all the possible outcomes, x . You can also determine the probability of any set of possible outcomes. A probability distribution table for a random variable, X , consists of a table with all the possible outcomes, along with the probability associated with each of the outcomes. The expected value and the variance of a probability distribution can be calculated using the following formulas:

$$E(x) = \mu_X = \sum xp(x)$$

$$\sigma^2_X = \sum (x - \mu_X)^2 p(x)$$

For the random variables X and Y and constants c and d , the mean and the standard deviation of a linear transformation are given by the following:

$$\mu_{c+dX} = c + d\mu_X$$

$$\sigma_{c+dX} = |d|\sigma_X$$

If the random variables X and Y are added or subtracted, the mean is calculated as shown below:

$$\mu_{X+Y} = \mu_X + \mu_Y$$

$$\mu_{X-Y} = \mu_X - \mu_Y$$

If X and Y are independent, then the following formulas can be used to compute the variance:

$$\sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y$$

$$\sigma^2_{X-Y} = \sigma^2_X + \sigma^2_Y$$

Points to Consider

- Are these concepts applicable to real-life situations?
- Will knowing these concepts allow you estimate information about a population?

Multimedia Links

For examples of finding means and standard deviations of sums and differences of random variables (5.0), see [mrjaffesclass, Linear Combinations of Random Variables](#) (6:41).

$$\begin{aligned}
 &X \mu_X = 2 \quad \sigma_X^2 = 3 \quad Y \mu_Y = 4 \quad \sigma_Y^2 = 4 \\
 &A = \frac{X+Y}{2} = \frac{1}{2}X + \frac{1}{2}Y \\
 &\mu_A = \frac{1}{2}\mu_X + \frac{1}{2}\mu_Y = \frac{1}{2}(2) + \frac{1}{2}(4) = 3 \\
 &\sigma_A^2 = \left(\frac{1}{2}\right)^2\sigma_X^2 + \left(\frac{1}{2}\right)^2\sigma_Y^2 = \frac{1}{4}(3) + \frac{1}{4}(4) = \frac{3}{4} + 1 = \frac{7}{4} \\
 &\sigma_A = \sqrt{\frac{7}{4}}
 \end{aligned}$$

MEDIA

Click image to the left or use the URL below.

URL: <http://www.ck12.org/flx/render/embeddedobject/1072>

Review Questions

1. It is estimated that 70% of the students attending a school in a rural area take the bus to school. Suppose you randomly select three students from the population. Construct the probability distribution of the random variable, X , defined as the number of students who take the bus to school. (Hint: Begin by listing all of the possible outcomes.)
2. The Safe Grad Committee at a high school is selling raffle tickets on a Christmas Basket filled with gifts and gift cards. The prize is valued at \$1200, and the committee has decided to sell only 500 tickets. What is the expected value of a ticket? If the students decide to sell tickets on three monetary prizes –one valued at \$1500 dollars and two valued at \$500 each, what is the expected value of the ticket now?
3. A recent law has been passed banning the use of hand-held cell phones while driving, and a survey has revealed that 76% of drivers now refrain from using their cell phones while driving. Three drivers were randomly selected, and a probability distribution table was constructed to record the outcomes. Let N represent those drivers who never use their cell phones while driving and S represent those who do use their cell phones while driving. Calculate the expected value and the variance using your calculator.

4.5 The Binomial Probability Distribution

Learning Objectives

- Know the characteristics of a binomial random variable.
- Understand a binomial probability distribution.
- Know the definitions of the mean, the variance, and the standard deviation of a binomial random variable.
- Identify the type of statistical situation to which a binomial distribution can be applied.
- Use a binomial distribution to solve statistical problems.

Introduction

Many experiments result in responses for which there are only two possible outcomes, such as either 'yes' or 'no', 'pass' or 'fail', 'good' or 'defective', 'male' or 'female', etc. A simple example is the toss of a coin. Say, for example, that we toss the coin five times. In each toss, we will observe either a head, H , or a tail, T . We might be interested in the probability distribution of X , the number of heads observed. In this case, the possible values of X range from 0 to 5. It is scenarios like this that we will examine in this lesson.

Binomial Experiments

Example: Suppose we select 100 students from a large university campus and ask them whether they are in favor of a certain issue that is being debated on their campus. The students are to answer with either a 'yes' or a 'no'. Here, we are interested in X , the number of students who favor the issue (a 'yes'). If each student is randomly selected from the total population of the university, and the proportion of students who favor the issue is p , then the probability that any randomly selected student favors the issue is p . The probability of a selected student who does not favor the issue is $1 - p$. Sampling 100 students in this way is equivalent to tossing a coin 100 times. This experiment is an example of a *binomial experiment*.

Characteristics of a Binomial Experiment

- The experiment consists of n independent, identical trials.
- There are only two possible outcomes on each trial: S (for success) or F (for failure).
- The probability of S remains constant from trial to trial. We will denote it by p . We will denote the probability of F by q . Thus, $q = 1 - p$.
- The binomial random variable X is the number of successes in n trials.

Example: In the following two examples, decide whether X is a binomial random variable.

Suppose a university decides to give two scholarships to two students. The pool of applicants is ten students: six males and four females. All ten of the applicants are equally qualified, and the university decides to randomly select two. Let X be the number of female students who receive the scholarship.

If the first student selected is a female, then the probability that the second student is a female is $\frac{3}{9}$. Here we have a conditional probability: the success of choosing a female student on the second trial depends on the outcome of the first trial. Therefore, the trials are not independent, and X is not a binomial random variable.

A company decides to conduct a survey of customers to see if its new product, a new brand of shampoo, will sell well. The company chooses 100 randomly selected customers and asks them to state their preference among the new shampoo and two other leading shampoos on the market. Let X be the number of the 100 customers who choose the new brand over the other two.

In this experiment, each customer either states a preference for the new shampoo or does not. The customers' preferences are independent of each other, and therefore, X is a binomial random variable.

Let's examine an actual binomial situation. Suppose we present four people with two cups of coffee (one percolated and one instant) to discover the answer to this question: "If we ask four people which is percolated coffee and none of them can tell the percolated coffee from the instant coffee, what is the probability that two of the four will guess correctly?" We will present each of four people with percolated and instant coffee and ask them to identify the percolated coffee. The outcomes will be recorded by using C for correctly identifying the percolated coffee and I for incorrectly identifying it. A list of the 16 possible outcomes, all of which are equally likely if none of the four can tell the difference and are merely guessing, is shown below:

TABLE 4.16:

Number Who Correctly Identify Percolated Coffee	Outcomes, C (correct), I (incorrect)	Number of Outcomes
0	$IIII$	1
1	$ICII IIC ICI CII$	4
2	$ICCI IICC ICIC CIIC CICI CCII$	6
3	$CICC ICC CCCI CCIC$	4
4	$CCCC$	1

Using the Multiplication Rule for Independent Events, you know that the probability of getting a certain outcome when two people guess correctly, such as $CICI$, is $(\frac{1}{2})(\frac{1}{2})(\frac{1}{2})(\frac{1}{2}) = (\frac{1}{16})$. The table shows six outcomes where two people guessed correctly, so the probability of getting two people who correctly identified the percolated coffee is $\frac{6}{16}$. Another way to determine the number of ways that exactly two people out of four people can identify the percolated coffee is simply to count how many ways two people can be selected from four people:

$${}_4C_2 = \frac{4!}{2!2!} = \frac{24}{4} = 6$$

In addition, a graphing calculator can also be used to calculate binomial probabilities.

By pressing **[2ND][DISTR]**, you can enter 'binompdf(4,0.5,2)'. This command calculates the binomial probability for k (in this example, $k = 2$) successes out of n (in this example, $n = 4$) trials, when the probability of success on any one trial is p (in this example, $p = 0.5$).

A binomial experiment is a probability experiment that satisfies the following conditions:

- Each trial can have only two outcomes—one known as a success, and the other known as a failure.
- There must be a fixed number, n , of trials.
- The outcomes of the trials must be independent of each other. The probability of each success doesn't change, regardless of what occurred previously.
- The probability, p , of a success must remain the same for each trial.

The distribution of the random variable X , where x is the number of successes, is called a *binomial probability distribution*. The probability that you get exactly $x = k$ successes is as follows:

$$P(x = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

where:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Let's return to the coffee experiment and look at the distribution of X (correct guesses):

TABLE 4.17:

k	$P(x = k)$
0	$\frac{1}{16}$
1	$\frac{4}{16}$
2	$\frac{6}{16}$
3	$\frac{4}{16}$
4	$\frac{1}{16}$

The expected value for the above distribution can be calculated as follows:

$$E(x) = (0) \left(\frac{1}{16} \right) + (1) \left(\frac{4}{16} \right) + (2) \left(\frac{6}{16} \right) + (3) \left(\frac{4}{16} \right) + (4) \left(\frac{1}{16} \right)$$

$$E(x) = 2$$

In other words, you would expect half of the four to guess correctly when given two equally-likely choices. $E(x)$ can be written as $(4) \left(\frac{1}{2} \right)$, which is equivalent to np .

For a random variable X having a binomial distribution with n trials and a probability of success of p , the expected value (mean) and standard deviation for the distribution can be determined by the following formulas:

$$E(x) = \mu_X = np \text{ and } \sigma_X = \sqrt{np(1-p)}$$

To apply the binomial formula to a specific problem, it is useful to have an organized strategy. Such a strategy is presented in the following steps:

- Identify a success.
- Determine p , the probability of success.
- Determine n , the number of experiments or trials.
- Use the binomial formula to write the probability distribution of X .

Example: According to a study conducted by a telephone company, the probability is 25% that a randomly selected phone call will last longer than the mean value of 3.8 minutes. What is the probability that out of three randomly selected calls:

- Exactly two last longer than 3.8 minutes?
- None last longer than 3.8 minutes?

Using the first three steps listed above:

- A success is any call that is longer than 3.8 minutes.
- The probability of success is $p = 0.25$.
- The number of trials is $n = 3$.

Thus, we can now use the binomial probability formula:

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

Substituting, we have: $p(x) = \binom{3}{x} (0.25)^x (1-0.25)^{3-x}$

a. For $x = 2$:

$$\begin{aligned} p(x) &= \binom{3}{2} (0.25)^2 (1-0.25)^{3-2} \\ &= (3)(0.25)^2 (1-0.25)^1 \\ &= 0.14 \end{aligned}$$

Thus, the probability is 0.14 that exactly two out of three randomly selected calls will last longer than 3.8 minutes.

b. Here, $x = 0$. Again, we use the binomial probability formula:

$$\begin{aligned} p(x=0) &= \binom{3}{0} (0.25)^0 (1-0.25)^{3-0} \\ &= \frac{3!}{0!(3-0)!} (0.25)^0 (0.75)^3 \\ &= 0.422 \end{aligned}$$

Thus, the probability is 0.422 that none of the three randomly selected calls will last longer than 3.8 minutes.

Example: A car dealer knows from past experience that he can make a sale to 20% of the customers who he interacts with. What is the probability that, in five randomly selected interactions, he will make a sale to:

- Exactly three customers?
- At most one customer?
- At least one customer?

Also, determine the probability distribution for the number of sales.

A success here is making a sale to a customer. The probability that the car dealer makes a sale to any customer is $p = 0.20$, and the number of trials is $n = 5$. Therefore, the binomial probability formula for this case is:

$$p(x) = \binom{5}{x} (0.2)^x (0.8)^{5-x}$$

a. Here we want the probability of exactly 3 sales, so $x = 3$.

$$p(x) = \binom{5}{3} (0.2)^3 (0.8)^{5-3} = 0.051$$

This means that the probability that the car dealer makes exactly three sales in five attempts is 0.051.

b. The probability that the car dealer makes a sale to at most one customer can be calculated as follows:

$$\begin{aligned}
 p(x \leq 1) &= p(0) + p(1) \\
 &= \binom{5}{0} (0.2)^0 (0.8)^{5-0} + \binom{5}{1} (0.2)^1 (0.8)^{5-1} \\
 &= 0.328 + 0.410 = 0.738
 \end{aligned}$$

c. The probability that the car dealer makes at least one sale is the sum of the probabilities of him making 1, 2, 3, 4, or 5 sales, as is shown below:

$$p(x \geq 1) = p(1) + p(2) + p(3) + p(4) + p(5)$$

We can now apply the binomial probability formula to calculate the five probabilities. However, we can save time by calculating the complement of the probability we're looking for and subtracting it from 1 as follows:

$$\begin{aligned}
 p(x \geq 1) &= 1 - p(x < 1) = 1 - p(x = 0) \\
 1 - p(0) &= 1 - \binom{5}{0} (0.2)^0 (0.8)^{5-0} \\
 &= 1 - 0.328 = 0.672
 \end{aligned}$$

This tells us that the salesperson has a probability of 0.672 of making at least one sale in five attempts.

We are also asked to determine the probability distribution for the number of sales, X , in five attempts. Therefore, we need to compute $p(x)$ for $x = 1, 2, 3, 4$, and 5. We can use the binomial probability formula for each value of X . The table below shows the probabilities.

TABLE 4.18:

x	$p(x)$
0	0.328
1	0.410
2	0.205
3	0.051
4	0.006
5	0.00032

Figure: The probability distribution for the number of sales.

Example: A poll of twenty voters is taken to determine the number in favor of a certain candidate for mayor. Suppose that 60% of all the city's voters favor this candidate.

- Find the mean and the standard deviation of X .
 - Find the probability of $x \leq 10$.
 - Find the probability of $x > 12$.
 - Find the probability of $x = 11$.
- a. Since the sample of twenty was randomly selected, it is likely that X is a binomial random variable. Of course, X here would be the number of the twenty who favor the candidate. The probability of success is 0.60, the percentage of the total voters who favor the candidate. Therefore, the mean and the standard deviation can be calculated as

shown:

$$\begin{aligned}\mu &= np = (20)(0.6) = 12 \\ \sigma^2 &= np(1-p) = (20)(0.6)(0.4) = 4.8 \\ \sigma &= \sqrt{4.8} = 2.2\end{aligned}$$

b. To calculate the probability that 10 or fewer of the voters favor the candidate, it's possible to add the probabilities that 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, or 10 of the voters favor the candidate as follows:

$$p(x \leq 10) = p(0) + p(1) + p(2) + \dots + p(10)$$

or

$$p(x \leq 10) = \sum_{x=0}^{10} p(x) = \sum_{x=0}^{10} \binom{20}{x} (0.6)^x (0.4)^{20-x}$$

As you can see, this would be a very tedious calculation, and it is best to resort to your calculator. Using a calculator (see the technology note below) with $n = 20$, $p = 0.6$, and $k \leq 10$, we get a probability of 0.245 that $x \leq 10$.

c. To find the probability that $x > 12$, it's possible to add the probabilities that 13, 14, 15, 16, 17, 18, 19, or 20 of the voters favor the candidate as shown:

$$p(x > 12) = p(13) + p(14) + \dots + p(20) = \sum_{x=13}^{20} p(x)$$

Alternatively, using the Complement Rule, $p(x > 12) = 1 - p(x \leq 12)$.

Using a calculator (see the technology note below) with $n = 20$, $p = 0.6$, and $k = 12$, we get a probability of 0.584 that $x \leq 12$. Thus, $p(x > 12) = 1 - 0.584 = 0.416$.

d. To find the probability that exactly 11 voters favor the candidate, it's possible to subtract the probability that less than or equal to 10 voters favor the candidate from the probability that less than or equal to 11 voters favor the candidate. These probabilities can be found using a calculator. Thus, the probability that exactly 11 voters favor the candidate can be calculated as follows:

$$p(x = 11) = p(x \leq 11) - p(x \leq 10) = 0.404 - 0.245 = 0.159$$

A graphing calculator will now be used to graph and compare different versions of a binomial distribution. Each binomial distribution will be entered into two lists and then displayed as a histogram. First, we will use the calculator to generate a sequence of integers, and next, we will use it to generate a corresponding list of binomial probabilities.

To generate a sequence of integers, press **[2ND][LIST]**, go to **OPS**, select '5:seq', enter '(X, X, 0, n, 1)', where n is the number of independent binomial trials, and press **[STO][2ND][L1]**.

To enter the binomial probabilities associated with this sequence of integers, press **[STAT]** and select '1:EDIT'.

Clear out **L2** and position the cursor on the **L2** list name.

Press [2ND][DISTR] to bring up the list of distributions.

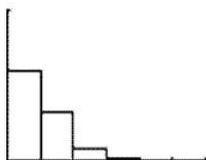
Select 'A:binompdf(' and enter ' n, p ', where n is the number of independent binomial trials and p is the probability of success.

To graph the histogram, make sure your window is set correctly, press [2ND][STAT PLOT], turn a plot on, select the histogram plot, choose L1 for Xlist and L2 for Freq, and press [GRAPH]. This will display the binomial histogram.

Horizontally, the following are examples of binomial distributions where n increases and p remains constant. Vertically, the examples display the results where n remains fixed and p increases.

$n = 5$ and $p = 0.1$

```
seq(X,X,0,5,1)→L1
{0 1 2 3 4 5}
binomPdf(5,.1)→L2
{.59049 .32805 ...
```



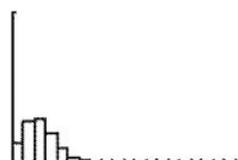
$n = 10$ and $p = 0.1$

```
seq(X,X,0,10,1)→L1
{0 1 2 3 4 5 6 ...}
binomPdf(10,.1)→L2
{.3486784401 .3...
```



$n = 20$ and $p = 0.1$

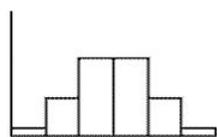
```
seq(X,X,0,20,1)→L1
{0 1 2 3 4 5 6 ...}
binomPdf(20,.1)→L2
{.1215766546 .2...
```



For a small value of p , the binomial distributions are skewed toward the higher values of X . As n increases, the skewness decreases and the distributions gradually move toward being more normal.

$n = 5$ and $p = 0.5$

```
seq(X,X,0,5,1)→L1
{0 1 2 3 4 5}
binomPdf(5,.5)→L2
{.03125 .15625 ...
```



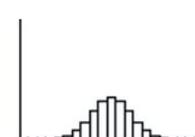
$n = 10$ and $p = 0.5$

```
seq(X,X,0,10,1)→L1
{0 1 2 3 4 5 6 ...}
binomPdf(10,.5)→L2
{9.765625E-4 .0...
```



$n = 20$ and $p = 0.5$

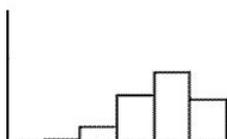
```
seq(X,X,0,20,1)→L1
{0 1 2 3 4 5 6 ...}
binomPdf(20,.5)→L2
{9.536743164E-7...
```



As p increases to 0.5, the skewness disappears and the distributions achieve perfect symmetry. The symmetrical, mound-shaped distribution remains the same for all values of n .

$n = 5$ and $p = 0.75$

```
seq(X,X,0,5,1)→L1
{0 1 2 3 4 5}
binomPdf(5,.75)→L2
{9.765625E-4 .0...
```



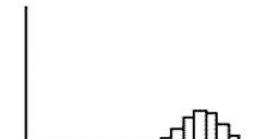
$n = 10$ and $p = 0.75$

```
seq(X,X,0,10,1)→L1
{0 1 2 3 4 5 6 ...}
binomPdf(10,.75)→L2
{9.536743164E-7...
```



$n = 20$ and $p = 0.75$

```
seq(X,X,0,20,1)→L1
{0 1 2 3 4 5 6 ...}
binomPdf(20,.75)→L2
{9.894947018E-1...
```



For a larger value of p , the binomial distributions are skewed toward the lower values of X . As n increases, the skewness decreases and the distributions gradually move toward being more normal.

Because $E(x) = np = \mu_X$, the expected value increases with both n and p . As n increases, so does the standard deviation, but for a fixed value of n , the standard deviation is largest around $p = 0.5$ and reduces as p approaches 0 or 1.

Technology Note: Calculating Binomial Probabilities on the TI-83/84 Calculator

Use the 'binompdf(' command to calculate the probability of exactly k successes. Press [2ND][DIST] and scroll down to 'A:binompdf('. Press [ENTER] to place 'binompdf(' on your home screen. Type values of n , p , and k , separated by commas, and press [ENTER].

Use the 'binomcdf(' command to calculate the probability of at most x successes. The format is 'binomcdf(n, p, k)' to find the probability that $x \leq k$. (Note: It is not necessary to close the parentheses.)

Technology Note: Using Excel

In a cell, enter the function =binomdist(x, n, p, false). Press [ENTER], and the probability of x successes will appear in the cell.

For the probability of at least x successes, replace 'false' with 'true'.

Lesson Summary

Characteristics of a Binomial Experiment:

- A binomial experiment consists of n identical trials.
- There are only two possible outcomes on each trial: S (for success) or F (for failure).
- The probability of S remains constant from trial to trial. We denote it by p . We denote the probability of F by q . Thus, $q = 1 - p$.
- The trials are independent of each other.
- The binomial random variable X is the number of successes in n trials.

The binomial probability distribution is: $p(x) = \binom{n}{x} p^x (1 - p)^{n-x} = \binom{n}{x} p^x q^{n-x}$.

For a binomial random variable, the mean is $\mu = np$.

The variance is $\sigma^2 = npq = np(1 - p)$.

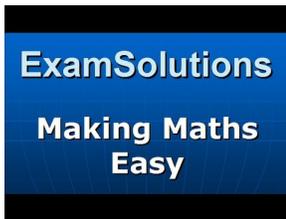
The standard deviation is $\sigma = \sqrt{npq} = \sqrt{np(1 - p)}$.

On the Web

<http://tinyurl.com/268m56r> Simulation of a binomial experiment. Explore what happens as you increase the number of trials.

Multimedia Links

For an explanation of binomial distribution and notation used for it **(4.0)(7.0)**, see [ExamSolutions, A-Level Statistics: Binomial Distribution \(Introduction\)](#) (10:31).

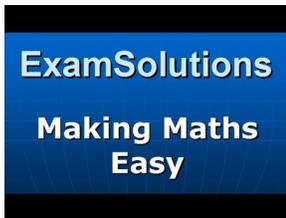


MEDIA

Click image to the left or use the URL below.

URL: <http://www.ck12.org/flx/render/embeddedobject/1073>

For an explanation on using tree diagrams and the formula for finding binomial probabilities **(4.0)(7.0)**, see [Exam Solutions, A-Level Statistics: Binomial Distribution \(Formula\)](#) (14:19).

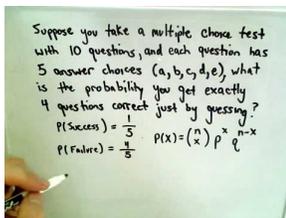


MEDIA

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URL: <http://www.ck12.org/flx/render/embeddedobject/1074>

For an explanation of using the binomial probability distribution to find probabilities **(4.0)**, see [patrickJMT, The Binomial Distribution and Binomial Probability Function](#) (6:45).



MEDIA

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URL: <http://www.ck12.org/flx/render/embeddedobject/1075>

Review Questions

- Suppose X is a binomial random variable with $n = 4$ and $p = 0.2$. Calculate $p(x)$ for each of the following values of X : 0, 1, 2, 3, 4. Give the probability distribution in tabular form.
- Suppose X is a binomial random variable with $n = 5$ and $p = 0.5$.
 - Display $p(x)$ in tabular form.
 - Compute the mean and the variance of X .
- Over the years, a medical researcher has found that one out of every ten diabetic patients receiving insulin develops antibodies against the hormone, thus, requiring a more costly form of medication.
 - Find the probability that in the next five patients the researcher treats, none will develop antibodies against insulin.
 - Find the probability that at least one will develop antibodies.
- According to the Canadian census of 2006, the median annual family income for families in Nova Scotia is \$56,400. [Source: Stats Canada. www.statcan.ca] Consider a random sample of 24 Nova Scotia households.
 - What is the expected number of households with annual incomes less than \$56,400?
 - What is the standard deviation of households with incomes less than \$56,400?
 - What is the probability of getting at least 18 out of the 24 households with annual incomes under \$56,400?

4.6 The Poisson Probability Distribution

Learning Objectives

- Know the definition of a Poisson distribution.
- Identify the characteristics of a Poisson distribution.
- Identify the type of statistical situation to which a Poisson distribution can be applied.
- Use a Poisson distribution to solve statistical problems.

Introduction

In this lesson, you will be introduced to Poisson distributions. Not only will you learn how to describe a Poisson distribution, but you will also learn how to apply the formula used with this type of distribution. Many real-world problems will be shown.

Poisson Distributions

A *Poisson probability distribution* is useful for describing the number of events that will occur during a specific interval of time or in a specific distance, area, or volume. Examples of such random variables are:

The number of traffic accidents at a particular intersection

The number of house fire claims per month that are received by an insurance company

The number of people who are infected with the AIDS virus in a certain neighborhood

The number of people who walk into a barber shop without an appointment

In a binomial distribution, if the number of trials, n , gets larger and larger as the probability of success, p , gets smaller and smaller, we obtain a Poisson distribution. The section below lists some of the basic characteristics of a Poisson distribution.

Characteristics of a Poisson Distribution

- The experiment consists of counting the number of events that will occur during a specific interval of time or in a specific distance, area, or volume.
- The probability that an event occurs in a given time, distance, area, or volume is the same.
- Each event is independent of all other events. For example, the number of people who arrive in the first hour is independent of the number who arrive in any other hour.

Poisson Random Variable

The probability distribution, mean, and variance of a Poisson random variable are given as follows:

$$p(x) = \frac{\lambda^x e^{-\lambda}}{x!} \quad x = 0, 1, 2, 3, \dots$$

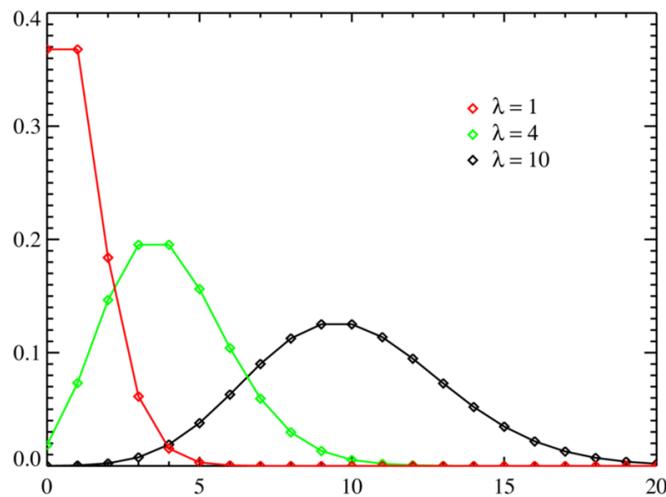
$$\mu = \lambda$$

$$\sigma^2 = \lambda$$

where:

λ = the mean number of events in the time, distance, volume, or area

e = the base of the natural logarithm



Example: A lake, popular among boat fishermen, has an average catch of three fish every two hours during the month of October.

- What is the probability distribution for X , the number of fish that you will catch in 7 hours?
 - What is the probability that you will catch 0 fish in seven hours of fishing? What is the probability of catching 3 fish? How about 10 fish?
 - What is the probability that you will catch 4 or more fish in 7 hours?
- a. The mean number of fish is 3 fish in 2 hours, or 1.5 fish/hour. This means that over seven hours, the mean number of fish will be $\lambda = 1.5 \text{ fish/hour} \bullet 7 \text{ hours} = 10.5 \text{ fish}$. Thus, the equation becomes:

$$p(x) = \frac{\lambda^x e^{-\lambda}}{x!} = \frac{(10.5)^x e^{-10.5}}{x!}$$

- b. To calculate the probabilities that you will catch 0, 3, or 10 fish, perform the following calculations:

$$p(0) = \frac{(10.5)^0 e^{-10.5}}{0!} \approx 0.000027 \approx 0\%$$

$$p(3) = \frac{(10.5)^3 e^{-10.5}}{3!} \approx 0.0053 \approx 0.5\%$$

$$p(10) = \frac{(10.5)^{10} e^{-10.5}}{10!} \approx 0.1236 \approx 12\%$$

This means that it is almost guaranteed that you will catch some fish in 7 hours.

c. The probability that you will catch 4 or more fish in 7 hours is equal to the sum of the probabilities that you will catch 4 fish, 5 fish, 6 fish, and so on, as is shown below:

$$p(x \geq 4) = p(4) + p(5) + p(6) + \dots$$

The Complement Rule can be used to find this probability as follows:

$$\begin{aligned} p(x \geq 4) &= 1 - [p(0) + p(1) + p(2) + p(3)] \\ &\approx 1 - 0.000027 - 0.000289 - 0.00152 - 0.0053 \\ &\approx 0.9929 \end{aligned}$$

Therefore, there is about a 99% chance that you will catch 4 or more fish within a 7 hour period during the month of October.

Example: A zoologist is studying the number of times a rare kind of bird has been sighted. The random variable X is the number of times the bird is sighted every month. We assume that X has a Poisson distribution with a mean value of 2.5.

- Find the mean and standard deviation of X .
 - Find the probability that exactly five birds are sighted in one month.
 - Find the probability that two or more birds are sighted in a 1-month period.
- a. The mean and the variance are both equal to λ . Thus, the following is true:

$$\begin{aligned} \mu &= \lambda = 2.5 \\ \sigma^2 &= \lambda = 2.5 \end{aligned}$$

This means that the standard deviation is $\sigma = 1.58$.

b. Now we want to calculate the probability that exactly five birds are sighted in one month. For this, we use the Poisson distribution formula:

$$\begin{aligned} p(x) &= \frac{\lambda^x e^{-\lambda}}{x!} \\ p(5) &= \frac{(2.5)^5 e^{-2.5}}{5!} \\ p(5) &= 0.067 \end{aligned}$$

c. The probability of two or more sightings is an infinite sum and is impossible to compute directly. However, we can use the Complement Rule as follows:

$$\begin{aligned} p(x \geq 2) &= 1 - p(x \leq 1) \\ &= 1 - [p(0) + p(1)] \\ &= 1 - \frac{(2.5)^0 e^{-2.5}}{0!} - \frac{(2.5)^1 e^{-2.5}}{1!} \\ &\approx 0.713 \end{aligned}$$

Therefore, according to the Poisson model, the probability that two or more sightings are made in a month is 0.713.

Technology Note: Calculating Poisson Probabilities on the TI-83/84 Calculator

Press [2ND][DIST] and scroll down to 'poissonpdf('. Press [ENTER] to place 'poissonpdf(' on your home screen. Type values of μ and x , separated by commas, and press [ENTER].

Use 'poissoncdf(' for the probability of at most x successes.

Note: It is not necessary to close the parentheses.

Technology Note: Using Excel

In a cell, enter the function =Poisson(μ, x, false), where μ and x are numbers. Press [ENTER], and the probability of x successes will appear in the cell.

For the probability of at least x successes, replace 'false' with 'true'.

Lesson Summary

Characteristics of a Poisson distribution:

- The experiment consists of counting the number of events that will occur during a specific interval of time or in a specific distance, area, or volume.
- The probability that an event occurs in a given time, distance, area, or volume is the same.
- Each event is independent of all other events.

Poisson Random Variable:

The probability distribution, mean, and variance of a Poisson random variable are given as follows:

$$p(x) = \frac{\lambda^x e^{-\lambda}}{x!} \quad x = 0, 1, 2, 3, \dots$$

$$\mu = \lambda$$

$$\sigma^2 = \lambda$$

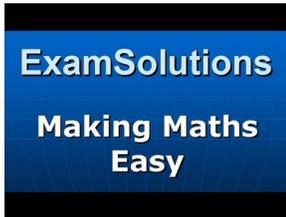
where:

λ = the mean number of events in the time, distance, volume or area

e = the base of the natural logarithm

Multimedia Links

For a discussion on the Poisson distribution and how to calculate probabilities (4.0)(7.0), see [ExamSolutions, Statistics: Poisson Distribution - Introduction](#) (12:32).



MEDIA

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URL: <http://www.ck12.org/flx/render/embeddedobject/1076>

For an example of finding probability in a Poisson situation (**7.0**), see [EducatorVids, Statistics: Poisson Probability Distribution](#) (1:55).



MEDIA

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URL: <http://www.ck12.org/flx/render/embeddedobject/1077>

Review Questions

A prison reports that the number of escape attempts per month has a Poisson distribution with a mean value of 1.5.

1. Calculate the probability that exactly three escapes will be attempted during the next month.
2. Calculate the probability that exactly one escape will be attempted during the next month.

4.7 Geometric Probability Distribution

Learning Objectives

- Know the definition of a geometric distribution.
- Identify the characteristics of a geometric distribution.
- Identify the type of statistical situation to which a geometric distribution can be applied.
- Use a geometric distribution to solve statistical problems.

Introduction

In this lesson, you will learn both the definition and the characteristics of a geometric probability distribution. In addition, you will be presented with a real-world problem that you can solve by applying what you have learned.

Geometric Probability Distributions

Like the Poisson and binomial distributions, a *geometric probability distribution* describes a discrete random variable. Recall that in the binomial experiments, we tossed a coin a fixed number of times and counted the number, X , of heads as successes.

A geometric distribution describes a situation in which we toss the coin until the first head (success) appears. We assume, as in the binomial experiments, that the tosses are independent of each other.

Characteristics of a Geometric Probability Distribution

- The experiment consists of a sequence of independent trials.
- Each trial results in one of two outcomes: success, S , or failure, F .
- The geometric random variable X is defined as the number of trials until the first S is observed.
- The probability $p(x)$ is the same for each trial.

Why would we wait until a success is observed? One example is in the world of business. A business owner may want to know the length of time a customer will wait for some type of service. Another example would be an employer who is interviewing potential candidates for a vacant position and wants to know how many interviews he/she has to conduct until the perfect candidate for the job is found. Finally, a police detective might want to know the probability of getting a lead in a crime case after 10 people are questioned.

Probability Distribution, Mean, and Variance of a Geometric Random Variable

The probability distribution, mean, and variance of a geometric random variable are given as follows:

$$p(x) = (1-p)^{x-1}p \quad x = 1, 2, 3, \dots$$

$$\mu = \frac{1}{p}$$

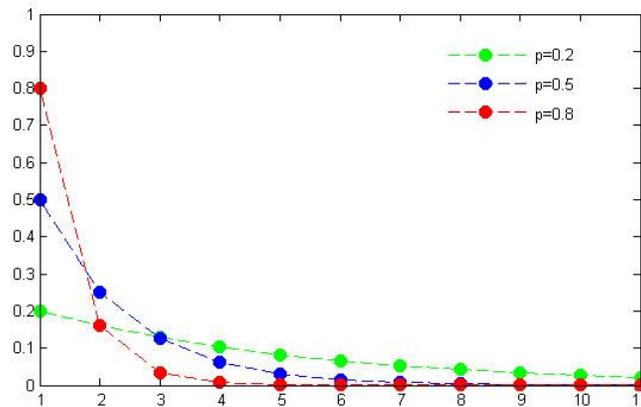
$$\sigma^2 = \frac{1-p}{p^2}$$

where:

p = probability of an S outcome

x = the number of trials until the first S is observed

The figure below plots a few geometric probability distributions. Note how the probabilities start high and drop off, with lower p values producing a faster drop-off.



Example: A court is conducting a jury selection. Let X be the number of prospective jurors who will be examined until one is admitted as a juror for a trial. Suppose that X is a geometric random variable, and p , the probability of a juror being admitted, is 0.50.

- Find the mean and the standard deviation of X .
 - Find the probability that more than two prospective jurors must be examined before one is admitted to the jury.
- a. The mean and the standard deviation can be calculated as follows:

$$\mu = \frac{1}{p} = \frac{1}{0.5} = 2$$

$$\sigma^2 = \frac{1-p}{p^2} = \frac{1-0.5}{0.5^2} = 2$$

$$\sigma = \sqrt{2} = 1.41$$

To find the probability that more than two prospective jurors will be examined before one is selected, you could try to add the probabilities that the number of jurors to be examined before one is selected is 3, 4, 5, and so on, as follows:

$$p(x > 2) = p(3) + p(4) + p(5) + \dots$$

However, this is an infinitely large sum, so it is best to use the Complement Rule as shown:

$$\begin{aligned} p(x > 2) &= 1 - p(x \leq 2) \\ &= 1 - [p(1) + p(2)] \end{aligned}$$

In order to actually calculate the probability, we need to find $p(1)$ and $p(2)$. This can be done by substituting the appropriate values into the formula:

$$\begin{aligned} p(1) &= (1 - 0.5)^{1-1}(0.5) = (0.5)^0(0.5) = 0.5 \\ p(2) &= (1 - 0.5)^{2-1}(0.5) = (0.5)^1(0.5) = 0.25 \end{aligned}$$

Now we can go back to the Complement Rule and plug in the appropriate values for $p(1)$ and $p(2)$:

$$\begin{aligned} p(x > 2) &= 1 - p(x \leq 2) \\ &= 1 - (0.5 + 0.25) = 0.25 \end{aligned}$$

This means that there is a 0.25 chance that more than two prospective jurors will be examined before one is admitted to the jury.

Technology Note: Calculating Geometric Probabilities on the TI-83/84 Calculator

Press **[2ND][DISTR]** and scroll down to 'geometpdf('. Press **[ENTER]** to place 'geometpdf(' on your home screen. Type in values of p and x separated by a comma, with p being the probability of success and x being the number of trials before you see your first success. Press **[ENTER]**. The calculator will return the probability of having the first success on trial number x .

Use 'geometcdf(' for the probability of at most x trials before your first success.

Note: It is not necessary to close the parentheses.

Lesson Summary

Characteristics of a Geometric Probability Distribution:

- The experiment consists of a sequence of independent trials.
- Each trial results in one of two outcomes: success, S , or failure, F .
- The geometric random variable X is defined as the number of trials until the first S is observed.
- The probability $p(x)$ is the same for each trial.

Geometric random variable:

The probability distribution, mean, and variance of a geometric random variable are given as follows:

$$\begin{aligned} p(x) &= (1 - p)^{x-1}p \quad x = 1, 2, 3, \dots \\ \mu &= \frac{1}{p} \\ \sigma^2 &= \frac{1 - p}{p^2} \end{aligned}$$

where:

p = probability of an S outcome

x = the number of trials until the first S is observed

Review Questions

1. The mean number of patients entering an emergency room at a hospital is 2.5. If the number of available beds today is 4 beds for new patients, what is the probability that the hospital will not have enough beds to accommodate its new patients?
2. An oil company has determined that the probability of finding oil at a particular drilling operation is 0.20. What is the probability that it would drill four dry wells before finding oil at the fifth one? (Hint: This is an example of a geometric random variable.)

Keywords

Addition Rule

Binomial experiment

Binomial probability distribution

Continuous

Continuous random variables

Discrete

Discrete random variables

Expected value

Geometric probability distribution

Linear Transformation Rule

Linear transformations

Poisson probability distribution

Probability distribution

Quantitative variables

Random variables

Re-centering

Rescale

Subtraction Rule